

A FREE BOUNDARY PROBLEM CONNECTED WITH THE OPTIMAL STOPPING PROBLEM FOR DIFFUSION PROCESSES

BY

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ABSTRACT. This paper deals with a free boundary problem for a parabolic equation in one space variable which arises from the problem of selecting an optimal stopping strategy for the diffusion process connected with the equation. It is shown that a solution of the free boundary problem yields the solution of a minimum problem concerning supersolutions of the parabolic equation as well as the solution of the optimal stopping problem. Theorems regarding the existence, uniqueness, regularity, and approach to the steady state of solutions of the free boundary problem are established.

1. Introduction. Let

$$L_0 \equiv a(x) \frac{\partial^2}{\partial x^2} + b(x) \frac{\partial}{\partial x} + c(x),$$

where $a, b, c \in C^{2+\alpha}([0, \infty))$ for some $\alpha \in (0, 1)$, $a(x) \geq a_0 > 0$ ($0 \leq x < \infty$), $c(x) \leq 0$ ($0 \leq x < \infty$), and let L be the parabolic operator $L = L_0 - \partial/\partial t$. Let $\psi \in C^3([0, \infty))$ be a given real-valued function such that, for some $d > 0$,

$$(1.1) \quad \begin{aligned} L_0 \psi &> 0 & (0 \leq x < d), \\ L_0 \psi &< 0 & (d < x < \infty). \end{aligned}$$

In this paper, we treat the following

Free boundary problem. Determine a curve $x = s(t)$ ($0 \leq t < \infty$) such that $s(t) > 0$, together with a solution u of the parabolic equation

$$(1.2) \quad Lu = 0$$

in the (unknown) region C defined by

$$C = \{(x, t); 0 < x < s(t), 0 < t < \infty\},$$

which satisfies the conditions

$$(1.3) \quad u(x, t) \geq \psi(x), \quad (x, t) \in C,$$

$$(1.4) \quad u(0, t) = \psi(0), \quad 0 \leq t < \infty,$$

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$$(1.5) \quad u(x, 0) = \psi(x), \quad 0 \leq x \leq s(0),$$

$$(1.6) \quad u(s(t), t) = \psi(s(t)), \quad 0 \leq t < \infty,$$

$$(1.7) \quad (\partial u / \partial x)(s(t), t) = \psi'(s(t)), \quad 0 \leq t < \infty.$$

Condition (1.3) will determine the value of $s(0)$; then the extra boundary condition will serve to determine uniquely the entire unknown portion of the boundary of C .

Our free boundary problem arises in connection with the following optimization problem in the theory of stochastic processes. Let $X(t)$ ($t \leq 0$) be a diffusion process on the nonnegative real axis with absorption at $x = 0$ and stopping at $t = 0$. Suppose the Dynkin generator of this process agrees on $(0, \infty)$ with the differential operator L_0 on smooth functions. Let \bar{M} be the set of all Markov times (stopping times) \bar{m} for this process. We wish to evaluate the function

$$v(x, t) = \sup_{\bar{m} \in \bar{M}} E_{X(-t)=x}(\psi(X(\bar{m})))$$

and to find a Markov time $\bar{m}_{\text{opt}} \in \bar{M}$ such that

$$E_{X(-t)=x}(\psi(X(\bar{m}_{\text{opt}}))) = v(x, t).$$

If we think of the elements of \bar{M} as stopping strategies, the problem is to find a strategy which will maximize the expected value of the *payoff function* ψ at the point of stopping, independently of the starting condition $X(-t) = x$ ($0 \leq t < \infty$). A strategy \bar{m}_{opt} which does this is called an *optimal stopping time*. It turns out that if we solve the free boundary problem (1.2)–(1.7) and set

$$(1.8) \quad \hat{u}(x, t) = \begin{cases} u(x, t) & \text{if } (x, t) \in C, \\ \psi(x) & \text{if } (x, t) \in \bar{Q} - C, \end{cases}$$

where $Q = (0, \infty) \times (0, \infty)$, then $v \equiv \hat{u}$ and we may choose

$$\bar{m}_{\text{opt}} = \inf \{t; (X(t), -t) \notin C\}.$$

I.e., the optimal strategy is to continue as long as the graph of position vs. time remaining lies in the *continuation region* C and to stop as soon as it leaves this region. The simple form of the optimal strategy is a consequence of the assumption (1.1) on the form of the payoff function, which has the interpretation that it pays to continue on a purely local basis when the sample path is in $(0, d)$ but it costs on a local basis when the sample path is in (d, ∞) . This interpretation may be read off from Dynkin's formula

$$E_x(\psi(X(\bar{m}))) = \psi(x) + E_x\left(\int_0^{\bar{m}} (L_0\psi)(X(t)) dt\right)$$

by setting

$$\bar{m} = \delta \wedge \inf\{t; |X(t) - x| \geq \delta\} \quad \text{for small } \delta > 0.$$

The connection between optimal stopping and free boundary problems was noticed by Chernoff [1] and McKean [13], among others, in special cases, and was explored in detail by Grigelionis and Širjaev [9] for general homogeneous Markov processes. We refer the reader to the survey papers of Chernoff [2] and Grigelionis [8] for an overview of the subject of optimal stopping problems and for a more extensive bibliography. The basic facts about optimal stopping may be found in Dynkin [3], [4] and Dynkin and Juškevič [5]. A number of free boundary problems which are similar to (1.2)–(1.7) have been treated by Ventcel' [19], Thi [18], Kruzhkov [12] and Sackett [15]. Kolodner [11], McKean [13] and Sherman [17] have reduced free boundary problems of optimal stopping type to nonlinear integral equations, but these have (so far) proved unamenable to solution. Also, Sackett [16] has solved numerically a free boundary problem which arises from an optimal stopping problem.

In §2, we assume that we have in hand a solution $\{s, u\}$ of (1.2)–(1.7) which has certain regularity and we establish a number of its properties. The main result (Theorem 1) is that, if \hat{u} is defined by (1.8), then \hat{u} is the *least of all L-supersolutions v on \bar{Q} such that $v \geq \psi$ on \bar{Q}* . This leads immediately to a uniqueness theorem (Theorem 2) for the free boundary problem. Next, we establish the connection described above between the free boundary problem and the optimal stopping problem (Theorem 3). Then we solve the steady state analogue of (1.2)–(1.7) (Theorem 4), which corresponds in the probabilistic interpretation to the case where the process is started at $t = -\infty$ (i.e., it is allowed to continue indefinitely until absorption occurs), and we establish the convergence of the solution of (1.2)–(1.7) to the steady state solution as $t \rightarrow \infty$ (Theorem 5). Finally, in §3 we construct the solution of the free boundary problem by an adaptation of the method of E. Röthe [14]. Here we make the mild technical assumption that

$$(1.9) \quad (L_0\psi)(x) \leq -c_0(x-d)^\beta \quad (x \geq d)$$

where c_0 is a positive constant and $0 < \beta \leq 1$. It turns out that $s(t)$ is Lipschitz for $0 < t < \infty$ and Hölder continuous on $[0, \infty)$ with exponent $\gamma = (1 + \beta)^{-1}$, and that \hat{u}_x and \hat{u}_t are Hölder continuous with exponent γ everywhere except at the origin.

The main tool used below is the *boundary point form of the maximum principle*:

if v is continuous and nonnegative in the closure of a domain D which is bounded below by $t = 0$ and on the right by a Lipschitz curve $x = s(t)$, and if $Lv \leq 0$ in D and $v(s(t_0), t_0) = 0$ for some $t_0 > 0$, then $v_x(s(t_0), t_0) < 0$ (cf., e.g., A. Friedman, *Remarks on the maximum principle for parabolic equations and its applications*, Pacific J. Math. 8 (1958), 201–211). In particular, the optimality theorems mentioned above follow from an application of this result. We remark here that, although the usual statements of this result require that D satisfy an "interior sphere condition" at $(s(t_0), t_0)$, it is a simple matter to achieve this by modifying the situation for $t > t_0$ without altering v for $t \leq t_0$ as long as $s(t)$ is Lipschitz.

2. Properties of the solution. Let $x = s(t)$ be a curve defined for $0 \leq t < \infty$ with $s(t) > 0$, and let u be a real-valued function defined on the open set

$$C = \{(x, t); 0 < x < s(t), 0 < t < \infty\}.$$

Definition 1. $\{s, u\}$ is a solution of (1.2)–(1.7) if

(i) $s \in C^\gamma([0, \infty))$ for some $\gamma \in [1/2, 1)$ and is Lipschitz continuous for $0 < t < \infty$.

(ii) $u \in C^{2,1}(C) \cap C^0(\bar{C})$.

(iii) $\partial u / \partial x$ is bounded and has a continuous extension to \bar{C} .

(iv) $\partial^2 u / \partial x^2$ is bounded and has a continuous extension to $\bar{C} - (0, 0) - (s(0), 0)$.

(v) Conditions (1.2)–(1.7) are satisfied.

We begin with a number of simple properties which solutions must satisfy.

Lemma 1. Let $\{s, u\}$ be a solution of (1.2)–(1.7). Then $s(0) = d$ and $s(t) \geq d$ for $0 < t < \infty$.

Proof. First suppose $s(0) > d$, and choose $x_0 \in (d, s(0))$. Then

$$\lim_{t \rightarrow 0} \frac{\partial u}{\partial t}(x_0, t) = \lim_{t \rightarrow 0} (L_0 u)(x_0, t) = L_0 \psi(x_0) < 0,$$

which contradicts (1.3). Hence, $s(0) \leq d$.

Now suppose there exists t_0 such that $s(t_0) < d$. Then there exist t_1, t_2 such that $0 \leq t_1 < t_2$ and $s(t) < d$ ($t_1 \leq t \leq t_2$). Let $D = \{(x, t); 0 < x < s(t), t_1 < t < t_2\}$, and define $v = u - \psi$ on \bar{D} . Then $v \geq 0$ on \bar{D} by (1.3) and $Lv \leq 0$ by (1.1) and (1.2). Moreover, $v(s(t), t) = 0$ ($t_1 \leq t \leq t_2$) by (1.6). But then we must have $v_x(s(t), t) < 0$ ($t_1 < t \leq t_2$) by the boundary point form of the maximum principle and this contradicts (1.7). Hence, $s(t) \geq d$ for all $t \geq 0$, which proves the lemma.

Lemma 2. Let $\{s, u\}$ be a solution of (1.2)–(1.7). Then $u_t > 0$ in C , and $s(t)$ is nondecreasing.

Proof. First observe that (1.6) implies that

$$u_x(s(t), t)s'(t) + u_t(s(t), t) = \psi'(s(t))s'(t)$$

for each $t > 0$ where $s(t)$ is differentiable. By (1.7), this implies $u_t(s(t), t) = 0$ for almost every $t > 0$. But then $u_t(s(t), t) \equiv 0$ by (iv) of Definition 1. We also have $u_t(0, t) = 0$ ($t > 0$) and $u_t(x, 0) \geq 0$ ($0 < x < d$) by (1.4) and (1.1) respectively. Furthermore, the derivatives u_{xx} , u_{xt} , u_{tt} exist and are (Hölder) continuous in C by an interior regularity theorem of Friedman [6], and hence $Lu_t \equiv 0$ in C . It follows that $u_t > 0$ in C by the strong maximum principle.

If $s(t)$ is not nondecreasing, we can find t_0 such that $s'(t_0)$ exists and is < 0 . Then $(s(t_0), t_0 - \delta) \in C$ for all sufficiently small positive δ . But then

$$\psi(s(t_0)) - u(s(t_0), t_0 - \delta) = \delta u_t(s(t_0), t_0 - \delta) > 0$$

for some $\delta' \in (0, \delta)$, and this contradicts (1.3). This completes the proof.

Remark. In view of the monotonicity of $s(t)$, the proof of Lemma 1 actually shows that $s(t) > d$ ($t > 0$).

We now characterize the solution as the solution of a minimum problem concerning supersolutions relative to the operator L .

Notation.

$$R_\delta(x_0, t_0) = \{(x, t); |x - x_0| < \delta, t_0 - \delta < t \leq t_0\},$$

$$\partial_p R_\delta(x_0, t_0) = \{(x, t) \in \partial R_\delta(x_0, t_0); t \neq t_0\}.$$

Definition 2. A function $v = v(x, t)$ is an L -supersolution if

- (i) v is defined and lower semicontinuous on \bar{Q} .
- (ii) For any $(x_0, t_0) \in Q$, there exists $\delta_0 > 0$ such that $R_{\delta_0}(x_0, t_0) \subseteq \bar{Q}$ and for all $\delta \in (0, \delta_0)$ we have that $v(x_0, t_0) \geq w(x_0, t_0)$ whenever $w \in C^{2,1}(R_\delta) \cap C^0(\bar{R}_\delta)$, $Lw = 0$ in R_δ , and $v \geq w$ on $\partial_p R_\delta$.

Theorem 1. Let $\{s, u\}$ be a solution of (1.2)–(1.7), and let \hat{u} be defined on \bar{Q} by (1.8). Then \hat{u} is the least L -supersolution on \bar{Q} which is $\geq \psi$ on \bar{Q} .

Proof. We have $\hat{u} \geq \psi$ by (1.3), and \hat{u} is continuous by (1.6). We show that \hat{u} is an L -supersolution. If (x_0, t_0) is in either the continuation region or the interior of its complement, condition (ii) of Definition 2 is verified for δ_0 equal to the distance of (x_0, t_0) from $\partial C \cup \{t = 0\}$ by application of the maximum principle. If $x_0 = s(t_0)$, $t_0 > 0$, we choose $\delta_0 > 0$ so small that the closure of $R_{\delta_0}(x_0, t_0)$ is contained in the open quarter plane $x > d$, $t > 0$. Define

$$D_{1,\delta} = R_\delta(x_0, t_0) \cap \{s(t) > x\}, \quad D_{2,\delta} = R_\delta(x_0, t_0) \cap \{s(t) < x\}$$

for $\delta \in (0, \delta_0)$. Let w be as in condition (ii) of Definition 2, and set $v = u - w$, $m = \inf_{\bar{R}_\delta} v$. Suppose $m < 0$. By the maximum principle, $v(x, t) \neq m$ for $(x, t) \in D_{1,\delta} \cup D_{2,\delta} \cup \partial_p R_\delta$. Therefore, there exists a point $(s(t_1), t_1)$, $t_0 - \delta < t_1 \leq t_0$, such that $v(s(t_1), t_1) = m$. By the boundary point form of the maximum principle,

$$v_x(s(t_1)-, t_1) < 0, \quad v_x(s(t_1)+, t_1) > 0.$$

But this contradicts (1.7) and the differentiability of w . Hence $m \geq 0$, which verifies condition (ii) of Definition 2.

Now let w be another L -supersolution such that $w \geq \psi$ in \bar{Q} . Set $v = w - u$ and, for any $T > 0$, let $m_T = \inf_{\bar{Q}_T} v$, where $Q_T = [0, \infty) \times (0, T]$. Assume $m_T < 0$. Since $v \geq 0$ outside C and v is lower semicontinuous, the set $S_T = \{(x, t) \in \bar{Q}_T; v(x, t) = m_T\}$ is a nonvoid closed set lying in C . Hence, there exists a point $(x_0, t_0) \in S_T$ such that $t_0 > 0$ and $(x, t) \notin S_T$ if $t < t_0$. If $\delta > 0$ is sufficiently small, we have $R_\delta(x_0, t_0) \subseteq C$. Let \tilde{u} be the solution of the first initial-boundary value problem

$$L\tilde{u} = 0 \quad \text{in } R_\delta(x_0, t_0), \quad \tilde{u} = u + m_T \quad \text{on } \partial_p R_\delta(x_0, t_0).$$

Since $\partial_p R_\delta \subseteq C - S_T$, we have $w > \tilde{u}$ on $\partial_p R_\delta$, and hence $w(x_0, t_0) \geq \tilde{u}(x_0, t_0)$ if δ is sufficiently small since w is an L -supersolution. But $\tilde{u} > u + m_T$ in R_δ by the strong maximum principle since

$$L(\tilde{u} - u - m_T) = -c(x)m_T \leq 0 \quad \text{in } R_\delta, \quad \tilde{u} - u - m_T = 0 \quad \text{on } \partial_p R_\delta.$$

Therefore, $v(x_0, t_0) > m_T$, which contradicts our choice of the point (x_0, t_0) . Hence, $m_T \geq 0$ and $w \geq \hat{u}$ in \bar{Q}_T . Therefore, $w \geq \hat{u}$ in \bar{Q} .

Theorem 2 (Uniqueness). *Let $\{s_1, u_1\}$ and $\{s_2, u_2\}$ be two solutions of (1.2)–(1.7). Then $s_1 \equiv s_2$ and $u_1 \equiv u_2$.*

Proof. Let C_1, C_2 be the continuation regions defined by s_1, s_2 respectively, and let \hat{u}_i ($i = 1, 2$) be defined as in (1.8). Then $u_1 \equiv u_2$ by Theorem 1. Suppose $s_1(t) < s_2(t)$ for some t . Then there is an open set $D \subseteq C_2 \cap (Q - C_1)$ in which we have

$$0 = Lu_2 = L\hat{u}_2 = L\hat{u}_1 = L\psi.$$

But this contradicts (1.1). Hence, $s_1 \equiv s_2$ and $u_1 \equiv u_2$.

We now establish the connection between the free boundary problem and the optimal stopping problem, which was described in § 1.

Theorem 3. *Let $\{s, u\}$ be a solution of (1.2)–(1.7), and let \hat{u} be defined as*

in (1.8). Let

$$v(x, t) = \sup_{\bar{m} \in \bar{M}} E_{X(-t)=x}(\psi(X(\bar{m})))$$

and

$$\bar{m}_{\text{opt}} = \inf\{t; (X(t), -t) \notin C\}.$$

Then

$$\hat{u}(x, t) = v(x, t) = E_{X(-t)=x}(\psi(X(\bar{m}_{\text{opt}})))$$

for all $(x, t) \in \bar{Q}$.

Proof. Since $\hat{u}(x, t) = E_{X(-t)=x}(\psi(X(\bar{m}_{\text{opt}})))$, we certainly have $u \leq v$ on \bar{Q} . The problem is to prove the reverse inequality.

If $\phi \in C_0^\infty(Q)$ and L^* is the formal adjoint of L , then

$$\begin{aligned} \int_Q \hat{u} L^* \phi &= \int_C u L^* \phi + \int_{Q-C} \psi L^* \phi \\ &= \int_C (Lu) \phi + \int_{Q-C} (L\psi) \phi = \int_Q (1 - \chi_C)(L\psi) \phi \end{aligned}$$

where χ_C is the characteristic function of C , for the boundary terms generated by the integrations by parts cancel by virtue of (1.6) and (1.7). Hence, $L\hat{u} = (1 - \chi_C)L\psi$ in Q in the distribution sense. By Duhamel's principle, \hat{u} may then be represented in the form

$$\hat{u}(x, t) = u_1(x, t) + \int_0^t w_r(x, t) dr$$

where u_1 is the solution of the problem $Lu_1 = 0$ in Q , $u_1 = \psi$ on ∂Q and $w_r(x, t)$ is defined for $r \leq t < \infty$ as the solution of the problem

$$\begin{aligned} Lw_r &= 0 \quad (x > 0, t > r), \\ w_r(0, t) &= 0 \quad (t > r), \\ w_r(x, r) &= (1 - \chi_C(x, r))(L\psi)(x, r). \end{aligned}$$

Now consider the Markov process with state space \bar{Q} whose sample paths starting at (x, t) are the backwards graphs $r \rightarrow (X(r-t), t-r)$ ($0 \leq r \leq t$) of the sample paths of the original process with starting condition $X(-t) = x$. The generator of this (space-time) process agrees on smooth functions with the differential operator L . The induced semigroup of operators on Borel functions defined on \bar{Q} is given by

$$\begin{aligned} H_r f(x, t) &= E_x f(X(r), t-r) \quad \text{if } 0 \leq r < t, \\ &= 0 \quad \text{if } r \geq t, \end{aligned}$$

and the corresponding Green operators are

$$G_\alpha f(x, t) = \int_0^t e^{-\alpha \tau} H_r f(x, t) d\tau \quad (\alpha \geq 0).$$

Set $f = (1 - \chi_C)(L\psi)$. Then $E_x f(X(r), t-r) = w_r(x, t)$, so that

$$G_0 f(x, t) = \int_0^t w_r(x, t) d\tau = \hat{u}(x, t) - u_1(x, t) \equiv \tilde{u}(x, t).$$

Hence, by Theorem 1 on p. 67 of [10], for all $\bar{m} \in \bar{M}$ we have

$$\begin{aligned} \tilde{u}(x, t) &= E_{X(-t)=x} \left(\int_{-t}^{\bar{m}} f(X(r), -r) d\tau \right) + E_{X(-t)=x} (\tilde{u}(X(\bar{m}), -\bar{m})) \\ &\geq E_{X(-t)=x} (\tilde{u}(X(\bar{m}), -\bar{m})) \end{aligned}$$

since $f \geq 0$. Since u_1 is smooth and $Lu_1 = 0$, we have

$$u_1(x, t) = E_{X(-t)=x} (u_1(X(\bar{m}), -\bar{m}))$$

for all $\bar{m} \in \bar{M}$ by Theorem 9 on p. 85 of [10]. Therefore,

$$\hat{u}(x, t) \geq E_{X(-t)=x} (\hat{u}(X(\bar{m}), -\bar{m})) \geq E_{X(-t)=x} (\psi(X(\bar{m})))$$

by (1.3). Taking the supremum over all $\bar{m} \in \bar{M}$, we arrive at the desired inequality.

We now turn to the steady state analogue of the free boundary problem (1.2)–(1.7). By comparing the steady and unsteady cases, we obtain *a priori* bounds on the solution of (1.2)–(1.7) and its limiting behavior as $t \rightarrow \infty$.

Theorem 4. *Let u_s denote the solution of the boundary value problem*

$$(2.1) \quad L_0 u_s = 0 \quad (0 < x < s), \quad u_s(0) = \psi(0), \quad u_s(s) = \psi(s).$$

Assume that

$$(2.2) \quad \limsup_{\xi \rightarrow \infty} [L_0 \psi](\xi) < 0.$$

Then there exists a unique value $s = s_\infty$ such that

$$(2.3) \quad u'_s(s) = \psi'(s).$$

Moreover, $s_\infty > d$ and $u_\infty \equiv u_{s_\infty} > \psi$ ($0 < x < s_\infty$).

Proof. We remark that the solution u_s of (2.1) always exists since $c(x) \leq 0$. Let L_0^* denote the formal adjoint of L_0 , and let W be the solution of the initial value problem $L_0^* W = 0$, $W(0) = 0$, $W'(0) = 1$. Since W cannot have a positive maximum, we have

$$(2.4) \quad W(\xi) > 0 \quad (0 < \xi < \infty), \quad \liminf_{\xi \rightarrow \infty} W(\xi) > 0.$$

Apply the Green identity

$$\int_0^s [(L_0 v)w - v(L_0^* w)] d\xi = [a(\xi)(v'(\xi)w(\xi) - v(\xi)w'(\xi)) + b(\xi)v(\xi)w(\xi)]_{\xi=0}^{\xi=s}$$

with $v = \psi - u_s$, $w = W$; the result is

$$(2.5) \quad a(s)W(s)[\psi'(s) - u'_s(s)] = \int_0^s [L_0 \psi](\xi)W(\xi) d\xi.$$

In view of (1.1), (2.2) and (2.4), there is a unique positive value of $s = s_\infty$ such that the right-hand side of (2.5) vanishes. This proves the existence of a unique solution $\{s_\infty, u_\infty\}$ of (2.1), (2.3). That $s_\infty > d$ is clear.

Since $u_\infty(s_\infty) = \psi(s_\infty)$ and $u'_\infty(s_\infty) = \psi'(s_\infty)$, it follows that

$$0 > [L_0(\psi - u_\infty)](s_\infty) = a(s_\infty)[\psi''(s_\infty) - u''_\infty(s_\infty)].$$

Hence, $u_\infty(x) > \psi(x)$ in some interval $s_\infty - \delta < x < s_\infty$ ($\delta > 0$). Let s_0 be the largest value of $x < s_\infty$ such that $u_\infty(x) = \psi(x)$. Then $0 \leq s_0 < d$, for otherwise the facts that $L_0(\psi - u_\infty) < 0$ ($s_0 < x < s_\infty$) and $\psi(x) = u(x)$ for $x = s_0$, s_∞ would imply that $\psi > u_\infty$ for $s_0 < x < s_\infty$ by the maximum principle. Likewise, if $0 < s_0 < d$, the maximum principle implies that $u_\infty(x) \geq \psi(x)$ for $0 < x < s_0$, so that $u_\infty - \psi$ has an interior minimum at $x = s_0$. Consequently, $u'_\infty(s_0) = \psi'(s_0)$. But this is impossible by (2.5) since $u_\infty \equiv u_{s_0}$ if $u_\infty(s_0) = \psi(s_0)$. Hence, $s_0 = 0$, which proves that $u_\infty > \psi$ in $0 < x < s_\infty$.

Theorem 5. Let $\{s, u\}$ be a solution of (1.2)–(1.7), and let $\{s_\infty, u_\infty\}$ be the solution of (2.1), (2.3). Then

$$(2.6) \quad s(t) < s_\infty \quad (0 \leq t < \infty), \quad u(x, t) < u_\infty \quad \text{in } C.$$

Moreover,

$$(2.7) \quad s(t) \rightarrow s_\infty, \quad u(x, t) \rightarrow u_\infty(x) \quad \text{uniformly as } t \rightarrow \infty.$$

Proof. It is easy to see that the function

$$\hat{u}_\infty(x) = \begin{cases} u_\infty(x), & 0 \leq x \leq s_\infty, \\ \psi(x), & s_\infty < x < \infty, \end{cases}$$

is an L -supersolution on \bar{Q} . Hence, $\hat{u}_\infty \geq u$ on C . Moreover, $s(t) < s_\infty$ (all $t \geq 0$) implies $u_\infty > u$ on C by the strong maximum principle. Assume that (2.6) fails. Then there exists a smallest time $t_0 > 0$ such that $s(t_0) = s_\infty$. Then $L(u_\infty - u) = 0$ in $0 < x < s(t)$, $0 < t \leq t_0$ and $(u_\infty - u)$ has a nonpositive minimum at (s_∞, t_0) . By the boundary point maximum principle, it follows that $(u_\infty - u)_x < 0$ at (s_∞, t_0) , which contradicts (1.7) and (2.3). Thus, (2.6) is proved.

Now we apply Green's formula

$$\iint_{C \cap Q_t} [(Lv)w - v(L^*w)] dx \wedge dt = \oint_{\partial(C \cap Q_t)} \{[d(v_x w - vw_x) + bvw] dt + vw dx\}$$

with $v = \psi - u$ and $w = W$ (W as in the proof of Theorem 4). The result is

$$(2.8) \quad \int_0^t \left[\int_0^{s(\tau)} (L_0 \psi)(\xi) W(\xi) d\xi \right] d\tau + \int_0^{s(t)} [\psi(\xi) - u(\xi, t)] W(\xi) d\xi = 0.$$

Since $s(t)$ and $u(x, t)$ are bounded above by (2.6) and nondecreasing in t by Lemma 1, the limits

$$s^\infty = \lim_{t \rightarrow \infty} s(t), \quad u^\infty(x) = \lim_{t \rightarrow \infty} u(x, t)$$

exist. Moreover, the second term in (2.8) is bounded. Since $s(\tau) > d$ is nondecreasing, the bracketed integral in the first term of (2.8) is nonincreasing. Since the first term must be bounded, this implies that

$$\lim_{t \rightarrow \infty} \int_0^{s(t)} [L_0 \psi](\xi) W(\xi) d\xi = \int_0^{s^\infty} [L_0 \psi](\xi) W(\xi) d\xi = 0.$$

Therefore, $s^\infty = s_\infty$.

Now let $\hat{u}(x, t)$ be defined by (1.8), and let U be the solution of $LU = 0$ in the half-strip $0 < x < s_\infty$, $t > 0$ which equals ψ on the boundary. Since \hat{u} is an L -supersolution, it may be shown that $\hat{u} \geq U$ in the half-strip (cf. the last paragraph of the proof of Theorem 1). But $u_\infty \geq \hat{u}$, and $U(x, t) \rightarrow u_\infty(x)$ uniformly in x by elementary methods, so that $\lim_{t \rightarrow \infty} \hat{u}(x, t) = u^\infty(x) = u_\infty(x)$, and the convergence is uniform.

3. Construction of the solution. We now construct a discrete-time analogue of problem (1.2)–(1.7) and prove that its solution converges to a solution of (1.2)–(1.7) as the time step tends to zero. The possibility of solving the approximate problem rests on the following lemma.

Lemma 3. *Let $\lambda > 0$ be given, and let*

$$M = \sup_{0 \leq x \leq s_\infty} |\psi(x)|.$$

Assume that $s \in [d, s_\infty)$ and $v \in C^2([0, s])$ are given such that $v \geq \psi$, $|v| \leq M$, $Lv \geq 0$ but $\neq 0$ on $[0, s]$, $v(0) = 0$, $v(s) = \psi(s)$, $v'(s) = \psi'(s)$. Let \hat{v} be defined on $[0, \infty)$ by

$$\hat{v}(x) = \begin{cases} v(x), & 0 \leq x \leq s, \\ \psi(x), & s < x < \infty. \end{cases}$$

Assume that (2.2) holds.

Then there exists a unique number $s' > s$ such that the solution u_σ of the problem

$$(3.1) \quad \begin{aligned} L_0 u_\sigma &= \lambda(u_\sigma - \hat{v}) \quad \text{on } (0, \sigma), \\ u_\sigma(0) &= \psi(0), \quad u_\sigma(\sigma) = \psi(\sigma) \end{aligned}$$

also satisfies

$$(3.2) \quad u'_\sigma(\sigma) = \psi'(\sigma)$$

exactly when $\sigma = s'$. Moreover, $s' < s_\infty$ and we have $|u| \leq M$, $u \geq \hat{v}$ and $L_0 u \geq 0$ but not $\equiv 0$ on $[0, s']$ where $u = u_{s'}$.

Proof. Let W be the solution of the initial value problem

$$(3.3) \quad L_0^* W - \lambda W = 0, \quad W(0) = 0, \quad W'(0) = 1.$$

Since W cannot have a positive maximum, it follows that

$$(3.4) \quad W(\xi) > 0 \quad (\xi > 0), \quad \liminf_{\xi \rightarrow \infty} W(\xi) > 0.$$

By (3.1) and (3.3), we have

$$\int_0^\sigma [(L_0 u_\sigma)W - u_\sigma(L_0^* W)] d\xi = - \int_0^\sigma \hat{v}(\xi) [L_0^* W](\xi) d\xi.$$

By applying Green's identity to both sides and substituting the boundary values of u_σ and \hat{v} , we arrive at the equation

$$(3.5) \quad a(\sigma)W(\sigma)[\psi'(\sigma) - u'_\sigma(\sigma)] = \int_0^s [L_0 v](\xi) W(\xi) d\xi + \int_s^\sigma [L_0 \psi](\xi) W(\xi) d\xi.$$

In view of (1.1), (2.2), (3.4) and the assumption about $L_0 v$ on $[0, s]$, the right-

hand side of (3.5) has a unique zero $\sigma = s' > s$. There follows the unique solvability of (3.1), (3.2).

Since $u(s') = \psi(s')$, $u'(s') = \psi'(s')$, we have

$$a(s')[u''(s') - \hat{v}''(s')] = [L_0(u - \hat{v})](s') = -[L_0\psi](s') > 0.$$

Hence, $u > \hat{v}$ in some neighborhood $s' - \delta < x < s'$ ($\delta > 0$). Suppose that the inequality $u > \hat{v}$ fails in the interval $[s, s']$. Let $s_0 = \sup\{x \in [s, s']; u(x) = \hat{v}(x)\}$. Then $u'(s_0) \geq \hat{v}'(s_0) = \psi'(s_0)$. Since $u_{s_0} \equiv u$ on $[0, s_0]$, it follows that the right side of (3.5) is nonpositive for $\sigma = s_0 < s'$. But this is impossible. Therefore, $u(x) > \hat{v}(x)$ ($s \leq x < s'$).

Now let $w = u - v$ on $[0, s]$. Then w satisfies

$$(L_0 - \lambda)w = -L_0v \leq 0, \quad w(0) = 0, \quad u(s) = u(s) - \psi(s) > 0.$$

Hence, $w \geq 0$ on $[0, s]$ by the maximum principle, and we have shown that $u \geq \hat{v}$ on $[0, s']$. By (3.1), we also have $L_0u \geq 0$ in $[0, s']$. We have already seen that $L_0u(s') > 0$, so $L_0u \neq 0$.

Next, suppose that $s' \geq s_\infty$. Let w_1 be the solution of (3.1) with $\sigma = s_\infty$. Then $\psi'(s_\infty) \geq w_1'(s_\infty)$ by (3.5) with $\sigma = s_\infty$. This implies that $w_1 \geq \hat{v}$ on $[0, s_\infty]$ by the same reasoning given above for u . Therefore, $L_0(u_\infty - w_1) = -\lambda(w_1 - \hat{v}) \leq 0$. But then $u_\infty - w_1$ must take its maximum value on $[0, s_\infty]$ at $x = s_\infty$, and furthermore $u_\infty'(s_\infty) = \psi'(s_\infty) < w_1'(s_\infty)$. But this is a contradiction. Hence, $s' < s_\infty$.

Finally, on $[0, s']$ we clearly have

$$M \geq \max[\psi(0), \psi(s_\infty)] \geq u_\infty \geq u \geq \hat{v} \geq \psi \geq -M$$

so that $|u| \leq M$ there.

We now construct the discrete time approximation to problem (1.2)–(1.7). Let $h > 0$ be a time step and set $s_0 = d$, $u_0(x) = \psi(x)$ ($0 \leq x \leq d$). Then define sequences $\{s_n\}$, $\{u_n(x)\}$ inductively by solving at each stage the problem

$$(3.6) \quad \begin{aligned} (u_n(x) - \hat{u}_{n-1}(x))/h &= [L_0u_n](x) \quad (0 < x < s_n) \\ u_n(0) &= \psi(0), \quad u_n(s_n) = \psi(s_n), \quad u_n'(s_n) = \psi'(s_n) \end{aligned}$$

for s_n and $u_n(x)$. Here

$$\hat{u}_n(x) = \begin{cases} u_n(x), & 0 \leq x \leq s_n, \\ \psi(x), & s_n < x < \infty. \end{cases}$$

This construction is possible by virtue of Lemma 3. Moreover, Lemma 3 guarantees that

- (i) $d = s_0 < s_1 < \dots < s_n < \dots < s_\infty$,
 (ii) $\hat{u}_n(x) \leq u_{n+1}(x)$ ($0 \leq x \leq s_{n+1}$), $n = 0, 1, 2, \dots$,
 (iii) $|u_n(x)| \leq M$ and $[L_0 u_n](x) \geq 0$ ($0 \leq x \leq s_n$; $n = 0, 1, 2, \dots$).

As the analysis of this scheme is rather lengthy, we break it up into a progression of lemmas.

Lemma 4. Let $\hat{u}_{n-1}(x) = \psi(x) - b[L_0 \psi](x)$ and $s_{-1} = d$. Define

$$q_n(x) = (u_n(x) - \hat{u}_{n-1}(x))/b \quad (0 \leq x \leq s_n)$$

for $n = 0, 1, 2, \dots$. Then there exist constants M_0 and M_1 , which depend only on ψ and on the coefficients of L , such that $0 \leq q_n(x) \leq M_0$ ($n = 0, 1, 2, \dots$), $|u'_n(x)| \leq M_1$ ($n = 0, 1, 2, \dots$).

Proof. Note that $q_n(x)$ satisfies the equations

$$(3.7) \quad L_0 q_n(x) = \begin{cases} [q_n(x) - q_{n-1}(x)]/b & \text{if } 0 \leq x \leq s_{n-1} \\ [q_n(x) - L_0 \psi(x)]/b & \text{if } s_{n-1} < x \leq s_n \end{cases} \quad (n = 1, 2, \dots)$$

$$q_0(x) = L_0 \psi(x),$$

and that $q_n(0) = q_n(s_n) = q'_n(s_n) = 0$ and $L_0 q_n(s_n) > 0$ for $n = 1, 2, \dots$. Hence, $q''_n(s_n) > 0$, from which it follows that $q_n(x) > 0$ in some neighborhood $s_n - \delta < x < s_n$. Therefore, $q_n(x)$ has a positive interior maximum at some point $x_0 \in (0, s_n)$. But we cannot have $s_{n-1} < x_0 < s_n$ since then

$$0 \geq L_0 q_n(x_0) = b^{-1}[L_0 u_n(x_0) - L_0 \psi(x_0)] > 0$$

by (3.7). Hence, $0 \leq x_0 \leq s_{n-1}$, whence $q_n(x_0) \leq q_{n-1}(x_0)$ by (3.7). Therefore,

$$0 \leq q_n(x) \leq \max_{0 \leq \xi \leq d} L_0 \psi(\xi) \equiv M_0 \quad (n = 0, 1, 2, \dots).$$

Next, observe that

$$\begin{aligned} a(x)u'_n(x) &= - \int_x^{s_n} L_0 u_n(\xi) d\xi + a(s_n)\psi'(s_n) + b(s_n)\psi(s_n) - b(x)u_n(x) \\ &\quad + \int_x^{s_n} u_n(\xi)c(\xi) d\xi. \end{aligned}$$

Therefore, $|u'_n(x)| \leq a_0^{-1}[A \max_{[0, s_\infty]} |\psi'| + (2B + s_\infty C)M + s_\infty M_0] \equiv M_1$, where A, B, C are the maxima of $|a|, |b|, |c|$ respectively on $[0, s_\infty]$.

The next lemma provides us with an inequality which guarantees the uniform Hölder continuity of the approximating free boundary curves and forces

their convergence. This aspect of the existence proof may be regarded as the most novel technical feature of the problem under consideration.

Lemma 5. Define, for $d \leq x < \infty$, a nonnegative strictly increasing function $G(x)$ by

$$G(x) = - \int_d^x (L_0 \psi)(\xi) d\xi.$$

Then, for each $T > 0$, there exists a constant $M_2(T)$, depending only on ψ and the coefficients of L , such that

$$0 < [G(s_n) - G(s_{n-1})]/b \leq M_2(T) \quad (n = 1, 2, \dots, [T/b]).$$

Proof. For $n = 0, 1, 2, \dots$, we introduce the functions

$$p_n(x) = a(x)q'_n(x) + b(x)q_n(x).$$

Since $q_n(x)$ is continuously differentiable on $[0, s_n]$, Green's formula yields

$$(3.8) \quad p_n(x) = - \int_x^{s_n} (L_0 q_n)(\xi) d\xi + \int_x^{s_n} c(\xi) q_n(\xi) d\xi.$$

By (3.7), it follows that

$$p_n(s_{n-1}) = - \int_{s_{n-1}}^{s_n} \frac{q_n(\xi) - L_0 \psi(\xi)}{b} d\xi + \int_{s_{n-1}}^{s_n} c(\xi) q_n(\xi) d\xi,$$

so that

$$(3.9) \quad p_n(s_{n-1}) \leq -(G(s_n) - G(s_{n-1}))/b.$$

The lemma will then be established if we can obtain a lower bound $p_n(x) \geq -M_2(T)$ (we remark that we cannot expect a similar upper bound since appropriate compatibility conditions at the origin are not satisfied).

Since (for $n \geq 1$) we have $p_n(s_n) = 0$ and

$$p'_n(x) = (q_n(x) - L_0 \psi(x))/b - c(x)q_n(x) > 0 \quad (s_{n-1} < x \leq s_n)$$

by (3.7) and (3.8), it follows that $p_n(x) < 0$ ($s_{n-1} < x < s_n$). $p_n(x)$ cannot have its minimum value in $(s_{n-1}, s_n]$. Nor can the minimum occur at $x = 0$, for $q_n(0) = 0$ and $q'_n(0) \geq 0$ (since $q_n \geq 0$), nor at $x = s_{n-1}$, for then

$$0 \geq p'_n(s_{n-1} - 0) = (q_n(s_{n-1}) - q_{n-1}(s_{n-1}))/b - c(s_{n-1})q_n(s_{n-1}),$$

which implies that $q_n(s_{n-1}) = q'_n(s_{n-1}) = p_n(s_{n-1}) = 0$ since $q_n \geq 0$. Hence,

$p_n(x)$ takes its negative minimum value at some point $x_0 \in (0, s_{n-1})$. On this interval, we have

$$\begin{aligned} \frac{p_n(x) - p_{n-1}(x)}{b} &= a(x) \frac{q'_n(x) - q'_{n-1}(x)}{b} + b(x) \frac{q_n(x) - q_{n-1}(x)}{b} \\ &= a(x)(L_0 q'_n)'(x) + b(x)(L_0 q_n)(x) \\ &= a(x)[p'_n(x) + c(x)q_n(x)]' + b(x)[p'_n(x) + c(x)q_n(x)] \\ &= (L_0 p_n)(x) + a(x)c'(x)q_n(x). \end{aligned}$$

Since $L_0 p_n(x_0) \geq 0$, it follows that

$$\begin{aligned} \inf_{[0, s_n]} p_n(x) &= p_n(x_0) \geq p_{n-1}(x_0) - AC'M_0 b \\ &\geq \inf_{[0, s_{n-1}]} p_{n-1}(x) - AC'M_0 b \end{aligned}$$

where $C' = \max_{[0, s_\infty]} |c'|$. By induction, we obtain

$$(3.10) \quad \inf_{[0, s_n]} p_n(x) \geq \inf_{[0, d]} p_0(x) - AC'M_0 T \equiv -M_2(T) \quad (n = 0, 1, 2, \dots, [T/b]).$$

This completes the proof.

Now let us define a curve $x = s^b(t)$ by setting $s^b(nb) = s_n$ ($n = 0, 1, 2, \dots$) and determining $s^b(t)$ for $nb < t < (n+1)b$ from the equation

$$G(s^b(t)) = \frac{t - nb}{b} G(s_{n+1}) + \frac{(n+1)b - t}{b} G(s_n).$$

Then, by the last lemma, the function $G(s^b(t))$ is Lipschitz continuous on every interval $0 \leq t \leq T$ with Lipschitz constant $M_2(T)$. Let C_b be the approximate continuation region

$$C_b = \{(x, t); 0 < x < s^b(t), 0 < t < \infty\}$$

and define a function $u^b(x, t)$ on C_b by

$$(3.11) \quad u^b(x, t) = \frac{t - nb}{b} u_{n+1}(x) + \frac{(n+1)b - t}{b} \hat{u}_n(x)$$

if $nb \leq t < (n+1)b$. We will show that $\{s^b, u^b\}$ converges to a solution $\{s, u\}$ of (1.2)–(1.7) as $b \rightarrow 0$. The necessary estimates are given in the next lemma.

Lemma 6. *Let $\{s^b, u^b\}$ be defined as above, and let (1.9) be satisfied. Then*

(a) *for each $T > 0$ there exists a constant $M_3(T)$ such that*

$$(3.12) \quad d \leq s^b(t) < s_\infty \quad (0 \leq t < \infty)$$

and

$$(3.13) \quad 0 \leq s^b(t_2) - s^b(t_1) \leq M_3(T)(t_2 - t_1)^\gamma \quad (0 \leq t_1 \leq t_2 \leq T),$$

where $\gamma = (1 + \beta)^{-1} \in [1/2, 1)$;

(b) for $nb < t < (n+1)b$,

$$(3.14) \quad 0 \leq \partial u^b(x, t)/\partial t = q_{n+1}(x) \leq M_0 \quad (0 \leq x \leq s^b(t));$$

(c) for all $(x, t) \in C_b$, $|u(x, t)| \leq M$ and

$$(3.15) \quad 0 \leq L_0 u^b \leq M_0;$$

(d) for all $(x, t) \in C_b$,

$$(3.16) \quad |\partial u^b/\partial x| \leq M_4,$$

where $M_4 = \max(M_1, \sup_{[d, s_\infty]} |\psi'|)$;

(e) for each $T > 0$ there exists a constant $M_5(T)$ such that for all $\delta > 0$ we have

$$(3.17) \quad \begin{aligned} 0 &\leq (u^b(x, t + \delta) - u^b(x, t))/\delta \\ &\leq M_5(T)(1 + 2b/\delta)[(s^b(t) - x) + M_3(T)(\delta + 2b)^\gamma] \end{aligned}$$

whenever $(x, t) \in C_b$ and $t + \delta \leq T - b$.

Proof. (a) Since G is strictly increasing, so are $s^b(t)$ and $G(s^b(t))$. Then

$$\begin{aligned} M_2(T) &\geq \frac{G(s^b(t_2)) - G(s^b(t_1))}{t_2 - t_1} = -\frac{1}{t_2 - t_1} \int_{s^b(t_1)}^{s^b(t_2)} (L_0 \psi)(\xi) d\xi \\ &\geq \frac{1}{t_2 - t_1} \int_{s^b(t_1)}^{s^b(t_2)} c_0(\xi - d)^\beta d\xi = \frac{c_0^\gamma}{t_2 - t_1} [(s^b(t_2) - d)^{1+\beta} - (s^b(t_1) - d)^{1+\beta}] \\ &\geq \frac{c_0^\gamma}{t_2 - t_1} [s^b(t_2) - s^b(t_1)]^{1+\beta} \geq 0. \end{aligned}$$

Setting $M_3(T) = (M_2(T)/c_0^\gamma)^\gamma$, we obtain (3.13). (3.12) is obvious.

(b)–(d) follow immediately from (3.11) and Lemma 4.

(e) Choose n and k such that

$$(n-1)b \leq t < nb < \dots < (n+k-1)b < t + \delta \leq (n+k)b.$$

Then by (3.14) we have

$$\begin{aligned} u^b(x, t + \delta) - u^b(x, t) \\ = (t - (n-1)b)q_n(x) + \sum_{j=1}^{k-1} q_{n+j}(x)b + (t + \delta - (n+k-1)b)q_{n+k}(x). \end{aligned}$$

Now $q_m(s_m) = 0$ and $q'_m(x) \geq a_0^{-1}[\inf p_m - BM_0]$, so that, by Lemma 5,

$$q_m(x) = - \int_x^{s_m} q'_m(\xi) d\xi \leq M_5(T)(s_m - x),$$

where $M_5(T) = a_0^{-1}(M_2(T) + BM_0)$. Hence,

$$\begin{aligned} u^b(x, t + \delta) - u^b(x, t) &\leq M_5(T) \left[(t - (n-1)b)(s_n - x) + \sum_{j=1}^{k-1} b(s_{n+j} - x) \right. \\ &\quad \left. + (t + \delta - (n+k-1)b)(s_{n+k} - x) \right] \\ &\leq M_5(T)(k+1)b(s_{n+k} - x) \leq M_5(T)(\delta + 2b)(s_{n+k} - x). \end{aligned}$$

But

$$s_{n+k} - s^b(t) \leq M_3(T)[(n+k)b - t]^\gamma \leq M_3(T)(\delta + 2b)^\gamma,$$

so that

$$u^b(x, t + \delta) - u^b(x, t) \leq M_5(T)(\delta + 2b)[(s^b(t) - x) + M_3(T)(\delta + 2b)^\gamma],$$

yielding (3.17).

Lemma 7. *Let the hypotheses of Lemma 6 be satisfied. Then there exists a sequence $b_m \rightarrow 0$, a continuous function $s(t)$ defined on $[0, \infty)$, and a function $u(x, t)$ which is defined on the closure of the set $C = \{(x, t) \in Q; 0 < x < s(t)\}$ and satisfies a uniform Lipschitz condition there, such that*

- (a) $s^{b_m}(t) \rightarrow s(t)$ uniformly on compact sets,
- (b) $s(t)$ satisfies the inequalities

$$(3.18) \quad d \leq s(t) \leq s_\infty \quad (0 \leq t < \infty)$$

and

$$(3.19) \quad 0 \leq s(t_2) - s(t_1) \leq M_3(T)(t_2 - t_1)^\gamma \quad \text{if } 0 \leq t_1 \leq t_2 \leq T,$$

- (c) if $\delta, \epsilon > 0$, there exists an integer m_0 such that $K_\delta \subseteq C_{b_m}$ and

$$|u^{b_m}(x, t) - u(x, t)| < \epsilon \quad \text{for } (x, t) \in K_\delta$$

whenever $m \geq m_0$, where $K_\delta = \{(x, t) \in \bar{Q}; 0 \leq x \leq s(t) - \delta, 0 \leq t \leq \delta^{-1}\}$.

Proof. By (3.12) and (3.13), the family $\{s^b(t)\}$ is uniformly bounded and equicontinuous on compact sets. Hence, by the Arzela-Ascoli theorem together with Cantor's diagonal process, a sequence $b_m \rightarrow 0$ may be chosen so that (a) holds. (b) then follows immediately from (3.12) and (3.13). It is also clear that $K_\delta \subseteq C_{b_m}$ if m is sufficiently large. For such m , (3.14) and (3.16) imply that $u^{b_m}(x, t)$ satisfies a Lipschitz condition on K_δ with Lipschitz constant $\max(M_0, M_4)$. Hence, by the Arzela-Ascoli theorem together with a diagonal process, we may replace $\{b_m\}$ with a subsequence, also denoted $\{b_m\}$, such that u^{b_m} converges uniformly on each K_δ to a limit function u which satisfies a Lipschitz condition on $\bigcup_{\delta > 0} K_\delta$. u may then be extended to the closure of C while preserving the Lipschitz condition.

Lemma 8. Let $\{s, u\}$ be as defined in Lemma 7. Then u satisfies (1.2) in C . Also, u satisfies (ii) of Definition 1.

Proof. By the standard theory of weak solutions, if we prove that

$$(3.20) \quad \iint_C u L^* \phi \, dx \, dt = 0, \quad \text{all } \phi \in C_0^\infty(C),$$

where L^* is the formal adjoint of L , it will follow that u satisfies (1.2) in C , together with (ii) of Definition 1. To prove (3.20), let $\delta > 0$ and consider values of b, T such that $\text{supp } \phi \subseteq K_\delta \cap \bar{Q}_{T-b} \subseteq K_{\delta/2} \subseteq \bar{C}_b$ and $2M_3(T)b^\gamma < \delta$. Then

$$\iint_C u^b L^* \phi \, dx \, dt = \sum_{n=0}^{N-1} \int_{nb}^{(n+1)b} \int_0^{s_n} (L u^b) \phi \, dx \, dt$$

where $N = [T/b]$, for $K_\delta \cap [s_n, \infty) \times [nb, (n+1)b] = \emptyset$. From (3.11) we obtain

$$\begin{aligned} & \iint_C u^b L^* \phi \, dx \, dt \\ &= \sum_{n=0}^{N-1} \int_0^{s_n} \int_{nb}^{(n+1)b} \left[L_0 u_n \left(n+1 - \frac{t}{b} \right) - L_0 u_{n+1} \left(n - \frac{t}{b} \right) - \frac{u_{n+1} - u_n}{b} \right] \phi(x, t) \, dt \, dx \\ &= \sum_{n=0}^{N-1} \int_0^{s_n} L_0 (u_n - u_{n+1}) \left[\int_{nb}^{(n+1)b} \left(n+1 - \frac{t}{b} \right) \phi(x, t) \, dt \right] dx \\ &= \sum_{n=0}^{N-1} \int_0^{s_{n-1}} L_0 u_n \int_{nb}^{(n+1)b} \left(n+1 - \frac{t}{b} \right) [\phi(x, t) - \phi(x, t-b)] \, dt \, dx \\ &\quad + \sum_{n=0}^{N-1} \int_{s_{n-1}}^{s_n} L_0 u_n \int_{nb}^{(n+1)b} \left(n+1 - \frac{t}{b} \right) \phi(x, t) \, dt \, dx \end{aligned}$$

where we set $\phi \equiv 0$ outside C . Therefore,

$$(3.21) \quad \left| \iint_C u^b L^* \phi \, dx \, dt \right| \leq \frac{1}{2} s_\infty M_0 T b \max |\phi_t| + M_0 M_3(T) b^\gamma \max |\phi|.$$

Now set $b = b_m$ with $m \rightarrow \infty$. Since $u^b \rightarrow u$ uniformly on $\text{supp } \phi$ the estimate (3.21) implies (3.20).

Lemma 9. *Let $\{s, u\}$ be as defined in Lemma 7. Then (1.3)–(1.6) are satisfied. Also, (1.7) is satisfied in the sense that*

$$(3.22) \quad \lim_{\delta \rightarrow 0} u_x(s(t) - \delta, t) = \psi'(s(t)) \quad (0 \leq t < \infty).$$

Proof. (1.3)–(1.5) are satisfied trivially since they are satisfied for each u^b . To prove (1.6), let $\epsilon > 0$ be arbitrary, and let $(x_0, t_0) \in C$. Choose $b = b_m$ with m so large that $(x_0, t_0) \in C_b$ and

$$|u(x_0, t_0) - u^b(x_0, t_0)| + |\psi(s^b(t_0)) - \psi(s(t_0))| < 2\epsilon/3.$$

Then

$$|u(x_0, t_0) - \psi(s(t_0))| < |u^b(x_0, t_0) - \psi(s^b(t_0))| + 2\epsilon/3.$$

If $t_0 = (n + \theta)b$, $\theta \in [0, 1)$, and if m is so large that $|x_0 - s^b(t_0)| \leq 2\delta$ where $\delta = |s(t_0) - x|$, then

$$\begin{aligned} |u^b(x_0, t_0) - \psi(s^b(t_0))| &\leq |\hat{u}_n(x) - \psi(s_n)|(1 - \theta) + |u_{n+1}(x_0) - \psi(s_{n+1})|\theta \\ &\quad + |\psi(s^b(t_0)) - (1 - \theta)\psi(s_n) - \theta\psi(s_{n+1})| \\ &\leq M_4\{(1 - \theta)[|x_0 - s_n| + |s^b(t_0) - s_n|] + \theta[|x_0 - s_{n+1}| + |s^b(t_0) - s_{n+1}|]\} \\ &\leq M_4[|x_0 - s^b(t_0)| + 2(1 - \theta)|s^b(t_0) - s_n| + 2\theta|s^b(t_0) - s_{n+1}|] \end{aligned}$$

by (3.11) and Lemma 6, whence

$$|u^b(x_0, t_0) - \psi(s(t_0))| \leq 2M_4(\delta + M_3(T)b^\gamma) \quad \text{if } 0 \leq t_0 \leq T.$$

If m is so large that $3M_3(T)b^\gamma < \epsilon$, we obtain $|u(x_0, t_0) - \psi(s(t_0))| < \epsilon + 2M_4\delta$, whence we conclude that

$$(3.23) \quad |u(x_0, t_0) - \psi(s(t_0))| \leq 2M_4\delta.$$

In view of the continuity of u on \bar{C} , this establishes (1.6).

In any compact subset of $[0, s(t_0))$, $u_x^b(x, t_0)$ is defined and continuous if b is sufficiently small. By (3.15) and (3.16), these functions are uniformly bounded

and equicontinuous there. Hence, by restricting b to a subsequence of $\{b_m\}$, we may arrange that $u_x^b(x, t_0) \rightarrow u_x(x, t_0)$ uniformly on compact subsets. By (3.15), u_x then satisfies a Lipschitz condition in the x -variable which is uniform on C . Hence, u_x may be extended to ∂C via its tangential boundary limits. Proceeding as above, we may derive an inequality similar to (3.23) which will establish (3.22).

Lemma 10. *Let $\{s, u\}$ be as defined in Lemma 7. Then*

- (a) $s(t)$ is Lipschitz continuous for $t > 0$.
- (b) If u_x is defined on \bar{C} as in the proof of Lemma 9, then u_x is continuous on \bar{C} . Moreover, (1.7) holds.
- (c) u_t , and therefore u_{xx} , has a continuous extension to $\bar{C} - (0, 0)$.

Proof. Since Lemmas 8 and 9 provide all the information required in the proof of Lemma 1 and $s(t)$ is nondecreasing, we have $s(t) > d$ for $0 < t < \infty$. If $t_0 > 0$, let $\epsilon = (s(t_0) - d)/2$ and choose $\delta > 0$ such that $s(t) \geq d + \epsilon$ for $t_0 - \delta \leq t \leq t_0 + \delta$. Let $C(\epsilon) = \inf_{\xi \geq d+\epsilon} [-L_0 \psi(\xi)] > 0$. Then, since $G(s^b(t))$ satisfies a Lipschitz condition with Lipschitz constant $M_2(T)$ on $0 \leq t \leq T$, the same is true of $G(s(t))$ and we have

$$\begin{aligned} M_2(T) &\geq \left| \frac{G(s(t)) - G(s(t_0))}{t - t_0} \right| = \frac{1}{|t - t_0|} \int_{s(t) \wedge t_0}^{s(t) \vee t_0} [-L_0 \psi(\xi)] d\xi \\ &\geq C(\epsilon) \left| \frac{s(t) - s(t_0)}{t - t_0} \right| \quad (t_0 - \delta \leq t \leq t_0 + \delta). \end{aligned}$$

Hence, $|s(t) - s(t_0)| \leq C(\epsilon)^{-1} M_2(T) (t - t_0)$ ($t_0 - \delta \leq t \leq t_0 + \delta$) which proves (a).

Now u_x and u_t are in the class $C^{2,1}(C)$ and therefore satisfy parabolic equations in C . By Lemma 9, u_x has continuous tangential boundary values up to the Lipschitz curve $x = s(t)$. But it is easy to show by the classical methods of Gevrey [7] that a solution of a parabolic equation which has continuous tangential boundary values up to a Lipschitz boundary curve $x = s(t)$ is continuous up to that curve. The rest of (b) follows from standard facts. To prove (c), let $b = b_m$ with $m \rightarrow \infty$ in (3.17). We obtain

$$0 \leq (u(x, t + \delta) - u(x, t)) / \delta \leq M_5(T) [(s(t) - x) + \delta^\gamma]$$

for $(x, t) \in \bar{C}$ with $0 \leq t \leq T$. Letting $\delta \rightarrow 0$ yields $0 \leq u_t(x, t) \leq M_5(T)(s(t) - x)$ in $\bar{C} \cap Q_T$. Hence, u_t has tangential boundary values which vanish identically on $x = s(t)$. (c) now follows in the same way as (b).

We summarize the results of Lemmas 7–10 as

Theorem 6. *Let (1.9) be satisfied. Then there exists a solution $\{s, u\}$ of problem (1.2)–(1.7). Moreover, if \hat{u} is defined by (1.8), then \hat{u}_t and \hat{u}_x are Hölder continuous except at the origin.*

The last statement of the theorem is obvious.

Remark. In view of the proof of Lemma 10(a), the assumption that (1.9) holds for all $x \geq d$ may be considerably weakened. It is necessary only to suppose that (1.9) holds for $d \leq x \leq d + \epsilon$ for some $\epsilon > 0$ and that (2.2) is satisfied.

Added in proof. Since the submission of this paper, the author has learned of two recent developments which should be mentioned here. (i) P. Van Moerbeke, in a thesis written at The Rockefeller University, has solved an optimal stopping-free boundary problem by treating an equivalent Volterra integral equation. His initial condition differs from our (1.5), and satisfies a compatibility assumption which makes the free boundary curve turn out to be Lipschitz near $t = 0$. He also treats a number of specific cases not covered by his general theorem. (ii) A. Bensoussan and J. L. Lions have discussed the equivalence of an optimal stopping problem for a diffusion on R^n with a *parabolic variational inequality*, which is a weak formulation of the minimum problem in our Theorem 1. They prove existence, uniqueness, etc. for the latter problem.

REFERENCES

1. H. Chernoff, *Sequential tests for the mean of a normal distribution*, Proc. Fourth Berkeley Sympos. Math. Statist. and Probability, vol. 1, Univ. of California Press, Berkeley, Calif., 1961, pp. 79–92.
2. ———, *Optimal stochastic control*, Sankhyā Ser. A 30 (1968), 221–252. MR 39 #2494.
3. E. B. Dynkin, *Markov processes*, Fizmatgiz, Moscow, 1963; English transl., Vols. I, II, Die Grundlehren der math. Wissenschaften, Bände 121, 122, Academic Press, New York; Springer-Verlag, Berlin, 1965. MR 33 #1886; 33 #1887.
4. ———, *The optimum choice of the instant for stopping a Markov process*, Dokl. Akad. Nauk SSSR 150 (1963), 238–240 = Soviet Math. Dokl. 4 (1963), 627–629. MR 27 #4278.
5. E. B. Dynkin and A. A. Juškevič, *Markov processes; Theorems and problems*, "Nauka", Moscow, 1967; English transl., Plenum Press, New York, 1969. MR 36 #6006; 39 #3585a.
6. A. Friedman, *Interior estimates for parabolic systems of partial differential equations*, J. Math. Mech. 7 (1958), 393–417. MR 21 #7362.
7. M. Gevrey, *Sur les équations aux dérivées partielles du type parabolique*, J. Math. 9 (6) (1913), 305–471.
8. B. I. Grigelionis, *Optimal stopping of stochastic processes*, Proc. Sixth Math. Summer School: Prob. Theory and Math. Statist. (Kačivili, 1968), Akad. Nauk Ukrain. SSR Inst. Mat., Kiev, 1969, pp. 176–210. (Russian) MR 43 #2761.
9. B. I. Grigelionis and A. N. Širjaev, *On the Stefan problem and optimal stopping*

rules for Markov processes, *Teor. Veroyatnost. i Primenen.* 11 (1966), 612–631 = *Theor. Probability Appl.* 11 (1966), 541–558. MR 35 #7538.

10. K. Itô, *Lectures on stochastic processes*, Tata Institute of Fundamental Research, Bombay 1961.

11. I. I. Kolodner, *Free boundary problem for the heat equation with applications to problems of change of phase. I. General method of solution*, *Comm. Pure Appl. Math.* 9 (1956), 1–31. MR 19, 285.

12. S. N. Kružkov, *On some problems with unknown boundaries for the heat conduction equation*, *Prikl. Mat. Meh.* 31 (1967), 1009–1014 = *J. Appl. Math. Mech.* 31 (1967), 1014–1024. MR 37 #6618.

13. H. P. McKean, Jr., *A free boundary problem for the heat equation arising from a problem of mathematical economics*, *Industrial Management Review* 6 (1965), 13–39; reprinted in *The Random Character of Stock Market Prices*, P. Cootner (ed.), M.I.T. Press, Cambridge, Mass., 1964, pp. 525–532.

14. E. Röthe, *Zweidimensionale parabolische Randwertaufgaben als Grenzfall eindimensionaler Randwertaufgaben*, *Math. Ann.* 102 (1930), 650–670.

15. G. G. Sackett, *An implicit free boundary problem for the heat equation*, *SIAM J. Numer. Anal.* 8 (1971), 80–96. MR 44 #654.

16. ———, *Numerical solution of a parabolic free boundary problem arising in statistical decision theory*, *Math. Comp.* 25 (1971), 425–434.

17. B. Sherman, *Some comments on free boundary problems for parabolic equations arising in statistical decision theory*, *Rocketdyne Res. Report* #66-24, 1966.

18. N. D. Thi, *On a free boundary problem for a parabolic equation*, *Vestnik Moskov. Univ.* 2 (1966), 40–54. (Russian)

19. T. D. Ventcel', *A free boundary problem for the heat equation*, *Dokl. Akad. Nauk SSSR* 131 (1960), 1000–1003 = *Soviet Math. Dokl.* 1 (1960), 358–361. MR 22 #8225.

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